Stability of Viscous Rotating Gravitating Streams in a Magnetic Field

Prem Kumar Bhatia and Ravi Prakash Mathur

Department of Mathematics, M. B. M. Engineering College, Faculty of Engineering,
Jai Narain Vyas University, Jodhpur, India

Reprint requests to Prof. P. K. B.; E-mail: pkbhatia123@yahoo.com

Z. Naturforsch. **61a,** 258 – 262 (2006); received January 23, 2006

We have studied the stability of two superposed viscous compressible gravitating streams rotating about an axis perpendicular to the direction of a horizontal magnetic field. For wave propagation parallel to the direction of the magnetic field the dispersion relation is derived by solving the linearized perturbation equations. Both the viscosity and rotation are found to suppress the instability of the system.

Key words: Rotation; Magnetic Field; Gravitating Streams; Perturbations; Instability.

1. Introduction

The study of the instability of gravitating media is important because of its relevance in the fragmentation and collapse of interstellar media.

Among others, Mouschovias [1, 2] and Mestel and Paris [3] have demonstrated the importance of the instability in a gravitating homogeneous static medium in the context of collapse and fragmentation in magnetic molecular clouds. Sengar [4] examined the effect of a magnetic field on superposed gravitating homogeneous streams while Sorker and Sarazin [5] have drawn attention to the importance of this instability problem of gravitating media in gravitational plasma filaments in cooling flows in galaxies and clusters of galaxies. Vranjes and Cadez [6] have studied the effect of radiative processes on the gravitational instability in a static medium.

Singh and Khare [7] investigated the instability in two gravitating inviscid streams under a uniform horizontal magnetic field and a uniform rotation about the vertical. Agarwal and Bhatia [8] studied this problem for inviscid streams rotating uniformly about the axis of the magnetic field. Shrivastava and Vaghela [9] have examined the combined influence of variable streams and radiation on the magnetogravitational instability in an interstellar medium. The velocity shear instability in hydrodynamics and plasmas under varying assumptions has been studied in recent years by several researchers. Among others, Benjamin and Bridges [10] have shown that the velocity shear instability in hydrodynamics admits of a canonical Hamiltonian formulation. Allah [11] has studied the instability of streams in the presence of the effects of heat and mass transfer, Luo et al. [12] have examined the effect of negatively charged dust on the parallel velocity shear instability in magnetized plasmas.

In astrophysical situations the instability of gravitating streams in a uniform horizontal magnetic field is interesting when the streams rotate about an axis in the horizontal plane which is perpendicular to the direction of the magnetic field. The authors [13] have recently studied this problem for inviscid streams. For more realistic situations the effect of viscosity of the streams on the growth rate of the unstable modes should be examined. This aspect forms the subject of the present investigation where we study the problem for streams of equal kinematic viscosities.

2. Perturbation Equations

Consider two superposed gravitating viscous streams occupying, respectively, the regions $z > 0$ and $z < 0$. The streams are assumed to be ideally conducting and permeated by a horizontal magnetic field along the x-axis. The streams are assumed to be of uniform densities, ρ_1 and ρ_2, and uniform viscosities, μ_1 and μ_2, moving with uniform speeds V_1 and V_2 along the direction of the magnetic field. The whole system rotates with a uniform angular velocity $\vec{\Omega}$ about an axis which is perpendicular to the magnetic field.

The relevant linearized perturbation equations are:

$$\rho_s\left[\frac{\partial}{\partial t}\vec{u}_s+\left(\vec{V}_s\cdot\nabla\right)\vec{u}_s\right]= -\nabla\delta p_s+\left(\nabla\times\vec{h}\right)\times\vec{H}+\rho_s\nabla\delta\phi_s+\mu_s\nabla^2\vec{u}_s +\frac{1}{3}\mu_s\nabla(\nabla\cdot\vec{u}_s)+2\rho_s\left(\vec{u}_s\times\vec{\Omega}\right), \quad (1)$$

$$\frac{\partial}{\partial t}\vec{h}_s+\left(\vec{V}\cdot\nabla\right)\vec{h}=\nabla\times\left(\vec{u}_s\times\vec{H}\right), \quad (2)$$

$$\nabla^2\delta\phi_s=-G\delta\rho_s, \quad (3)$$

$$\frac{\partial}{\partial t}\delta p_s+\left(\vec{V}_s\cdot\nabla\right)\delta p_s=C_s^2\left[\frac{\partial}{\partial t}\delta\rho_s+\left(\vec{V}_s\cdot\nabla\right)\delta\rho_s\right], \quad (4)$$

$$\frac{\partial}{\partial t}\delta\rho_s+\left(\vec{V}_s\cdot\nabla\right)\delta\rho_s+\rho_s(\nabla\cdot\vec{u}_s)=0, \quad (5)$$

$$\nabla\cdot\vec{h}=0. \quad (6)$$

In these equations $\vec{h}=(h_x,h_y,h_z)$, $\delta\rho$, $\delta\phi$ and δp are the perturbations, respectively, in the magnetic field $\vec{H}$, density ρ, gravitational potential ϕ and pressure p due to the small disturbance given to the system which produces the velocity field $\vec{u}=(u,v,w)$ in the system. Equations (1) to (6) are the same for the two streams, and the subscript 's' distinguishes the two streams; $s=1$ corresponds to the upper region $z>0$ and $s=2$ to the lower region $z<0$.

The stability analysis for the considered configuration must be investigated for the modes of propagation along and perpendicular to the direction of the magnetic field, as has been studied by the authors recently [13] for inviscid streams. In the case of viscous superposed gravitating streams rotating about an axis perpendicular to the magnetic field, the analysis for the mode of propagation along the axis about which the system rotates, is not tractable mathematically.

For the considered configuration we, therefore, analyse mathematically the stability problem for the mode of wave propagation having a finite component parallel to the magnetic field. We assume that all the perturbed quantities vary in space and time as

$$F(z)\exp(\mathrm{i}k_x x+nt), \quad (7)$$

where $F(z)$ is some function of z, k_x is the wave number of the perturbation along the x-axis, and n (may be complex) is the rate at which the system departs away from the equilibrium. For perturbations of the form (7), equations (1) to (6) become, on writing $D\equiv\frac{\mathrm{d}}{\mathrm{d}z}$:

$$\rho_s\left[\sigma_s-\gamma_s\left(D^2-\frac{4}{3}k_x^2\right)\right]u_s= -\mathrm{i}k_x\delta p_s+\mathrm{i}k_x\rho_s\delta\phi_s-2\rho_s\Omega w_s+\frac{1}{3}\mathrm{i}k_x\rho_s\gamma_s D w_s, \quad (8)$$

$$\rho_s\left[\sigma_s-\gamma_s(D^2-k_x^2)\right]v_s=\mathrm{i}k_x h_y H_s, \quad (9)$$

$$\rho_s\left[\sigma_s-\gamma_s\left(\frac{4}{3}D^2-k_x^2\right)\right]w_s= -D\delta p_s+\rho_s D\delta\phi_s+H_s(\mathrm{i}k_x h_z-Dh_x) +2\rho_s\Omega u_s+\frac{1}{3}\rho_s\gamma_s D(\mathrm{i}k_x u_s), \quad (10)$$

$$\sigma_s(h_x,h_y,h_z)=H_s(-Dw_s,\mathrm{i}k_x v_s,\mathrm{i}k_x w_s), \quad (11)$$

$$(D^2-k_x^2)\delta\phi_s=-G\delta\rho_s, \quad (12)$$

$$\sigma_s\delta p_s=C_s^2\sigma_s\delta\rho_s, \quad (13)$$

$$\sigma_s\delta\rho_s=-\rho_s(\mathrm{i}k_x u_s+Dw_s), \quad (14)$$

$$\mathrm{i}k_x h_x+Dh_z=0, \quad (15)$$

where we have written

$$\sigma_s=n+\mathrm{i}k_x V_s. \quad (16)$$

On eliminating some of the variables from the above equations we finally obtain the sixth-order differential equation in $\delta\phi$:

$$(D^2-k_x^2)(D^2-\alpha_s^2)(D^2-\beta_s^2)\delta\phi_s=0, \quad (17)$$

where

$$\alpha_s^2+\beta_s^2=\frac{1}{N_s}\left[\sigma_s^2(M_s^2+C_s^2)+M_s^2C_s^2k_x^2+\gamma_s\sigma_s\left\{\frac{7}{3}\sigma_s^2+\frac{8}{3}\sigma_s\gamma_s k_x^2+\left(\frac{7}{3}M_s^2+2C_s^2\right)k_x^2-G\rho_s\right\}\right], \quad (18)$$

$$\alpha_s^2\beta_s^2=\frac{1}{N_s}\left[\begin{matrix}\sigma_s^4+\sigma_s^2\left\{(M_s^2+C_s^2)k_x^2-G\rho_s+4\Omega^2\right\}+M_s^2k_x^2(C_s^2k_x^2-G\rho_s)+\\ \sigma_s\gamma_s k_x^2\left\{\frac{7}{3}\sigma_s^2+\frac{4}{3}\sigma_s\gamma_s k_x^2+\left(\frac{4}{3}M_s^2+C_s^2\right)k_x^2-G\rho_s\right\}\end{matrix}\right], \quad (19)$$

$$N_s = \sigma_s \gamma_s \left(M_s^2 + C_s^2 + \frac{4}{3}\sigma_s \gamma_s \right). \tag{20}$$

3. The Superposed Streams

Now we seek the solutions of (17) which remain bounded in the regions occupied by the streams. The solutions appropriate to the two streams are therefore

$$\delta\phi_1 = A_1 \exp(-\alpha_1 z) + B_1 \exp(-\beta_1 z) + E_1 \exp(-k_x z) \quad (z > 0), \tag{21}$$

$$\delta\phi_2 = A_2 \exp(\alpha_2 z) + B_2 \exp(\beta_2 z) + E_2 \exp(k_x z)(z < 0), \tag{22}$$

where A_1, B_1, E_1, A_2, B_2, E_2 are constants of integration to be determined by applying appropriate boundary conditions to the solutions. In writing the solutions for $\delta\phi_1$ and $\delta\phi_2$ it is assumed that α_1, β_1, α_2, β_2 are so defined that their real parts are positive. The six boundary conditions which must be satisfied at the interface $z = 0$ are:

(i) continuity of the perturbed gravitational potential, i.e. $\delta\phi_1 = \delta\phi_2$;

(ii) continuity of the normal derivative of the perturbed gravitational potential, i. e. $D(\delta\phi_1) = D(\delta\phi_2)$;

(iii) continuity of the total perturbed pressure, i.e. $\delta p_1 + H(h_x)_1 = \delta p_2 + H(h_x)_2$;

(iv) uniqueness of the normal displacement at any point (fluid element) on the interface, i. e. $\frac{w_1}{\sigma_1} = \frac{w_2}{\sigma_2}$;

(v) continuity of the normal derivative of the displacement, i. e. $D\left(\frac{w_1}{\sigma_1}\right) = D\left(\frac{w_2}{\sigma_2}\right)$;

(vi) continuity of the tangential component of viscous stresses, i. e. $\mu_1(D^2 + k_x^2)\left(\frac{w_1}{\sigma_1}\right) = \mu_2(D^2 + k_x^2)\left(\frac{w_2}{\sigma_2}\right)$.

Now, on eliminating u_s, δp_s, h_x, and h_z, (10) can be written as

$$\frac{w_s}{\sigma_s} = \frac{\rho_s k_x^2 G + \left\{ C_s^2 k_x^2 + \sigma_s^2 + \frac{1}{3}\gamma_s \sigma_s k_x^2 - \gamma_s \sigma_s (D^2 - k_x^2) \right\} (D^2 - k_x^2)\delta\phi_s}{\rho_s \sigma_s G \left[\sigma_s D - \gamma_s (D^2 - k_x^2) D - 2\mathrm{i} k_x \Omega \right]}. \tag{23}$$

On applying the solutions (21) and (22) and using (23) the six boundary conditions lead to the following six relations:

$$A_1 + B_1 + E_1 - A_2 - B_2 - E_2 = 0, \tag{24}$$

$$\alpha_1 A_1 + \beta_1 B_1 + k_x E_1 + \alpha_2 A_2 + \beta_2 B_2 + k_x E_2 = 0, \tag{25}$$

$$T_1 A_1 + T_2 B_1 - \rho_1 G M_1^2 k_x^5 E_1 + T_3 A_2 + T_4 B_2 + \rho_2 G M_2^2 k_x^5 E_2 = 0, \tag{26}$$

$$Q_1 A_1 + Q_2 B_1 + S_1 E_1 + Q_3 A_2 + Q_4 B_2 + S_2 E_2 = 0, \tag{27}$$

$$\alpha_1 Q_1 A_1 + \beta_1 Q_2 B_1 + k_x S_1 E_1 + \alpha_2 Q_3 A_2 + \beta_2 Q_4 B_2 + k_x S_2 E_2 = 0, \tag{28}$$

$$\rho_1\gamma_1[(\alpha_1^2 + k_x^2)Q_1 A_1 + (\beta_1^2 + k_x^2)Q_2 B_1 + 2k_x^2 S_1 E_1] + \rho_2\gamma_2[(\alpha_2^2 + k_x^2)Q_3 A_2 + (\beta_2^2 + k_x^2)Q_4 B_2 + 2k_x^2 S_2 E_2] = 0, \tag{29}$$

where

$$T_1 = (\alpha_1^2 - k_x^2)\left[\sigma_1^2 \alpha_1^2 C_1^2 k_x^2 + M_1^2 C_1^2 k_x^4 + \alpha_1 \gamma_1 \sigma_1 k_x^4 \left(C_1^2 - \frac{1}{3}\alpha_1^2 M_1^2 \right) - 2\mathrm{i}\alpha_1^2 k_x \sigma_1 \Omega (M_1^2 + C_1^2)\right] - G\rho_1 M_1^2 k_x^2 \alpha_1^3, \tag{30}$$

$$Q_1 = \rho_2\sigma_2[-\sigma_2\alpha_1 + \gamma_2\alpha_1(\alpha_1^2 - k_x^2) - 2\mathrm{i}k_x\Omega]\left[\rho_1 G k_x^2 + \left\{\sigma_1^2 + \left(C_1^2 + \frac{1}{3}\gamma_1\sigma_1\right)k_x^2\right\}(\alpha_1^2 - k_x^2) - \sigma_1\gamma_1(\alpha_1^2 - k_x^2)^2\right], \tag{31}$$

$$S_1 = \sigma_2\rho_1\rho_2 G k_x^3(-\sigma_2 - 2\mathrm{i}\Omega). \tag{32}$$

The coefficient T_2 is obtained by changing α_1 to β_1 in T_1. T_3 is obtained from T_1 by changing σ_1 to σ_2, α_1 to α_2, C_1 to C_2, M_1 to M_2, ρ_1 to ρ_2, γ_1 to γ_2 and G to $-G$. The coefficient T_4 is obtained from T_3 by changing α_2 to β_2. The coefficient Q_2 is obtained from Q_1 by changing α_1 to α_2, i to $-$i and interchanging ρ_1 and ρ_2,

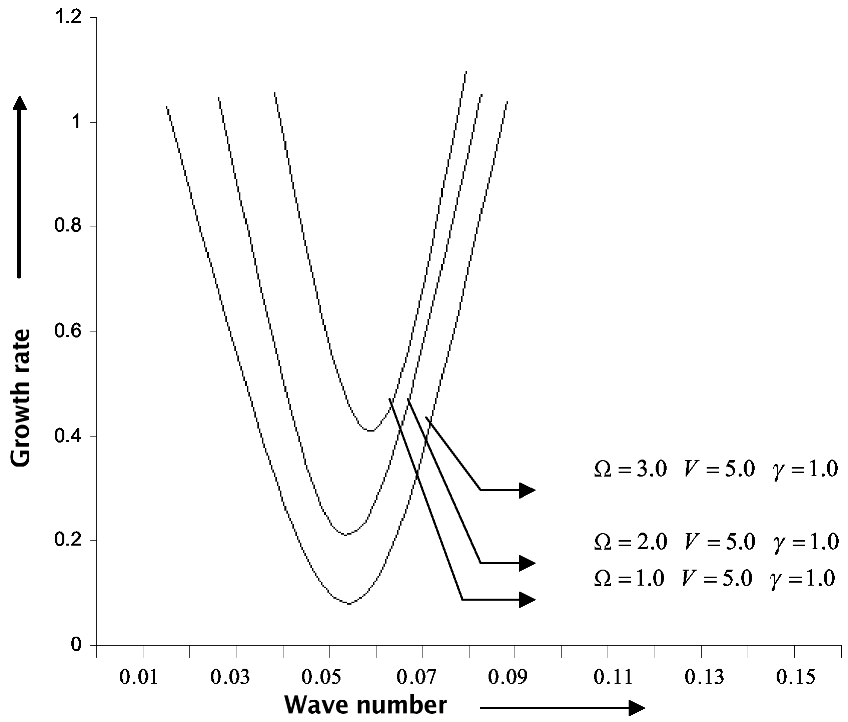

Fig. 1. Plot of the growth rate against the wave number.

σ_1 and σ_2, γ_1 and γ_2. On changing α_2 to β_2 in Q_3, we get Q_4. The coefficient S_2 is obtained from S_1 by changing σ_2 to σ_1 and i to $-$i.

4. The Dispersion Relation

For non-trivial solution the determinant of the matrix of the coefficients of $A_1, B_1, E_1, A_2, B_2, E_2$ in (24) to (29) must vanish. This gives the dispersion relation. Since the dispersion relation is quite complicated, it is not possible to solve it analytically. One has therefore to solve it numerically to ascertain the influence of the effects of rotation and viscosity on the instability of the system. As our objective is to determine qualitatively the effect of viscosity and rotation on the growth rate of the unstable mode of disturbance, and as the dispersion relation is quite complicated, we consider a simple model of the superposed streams. We consider the case of two streams of the same density and the same kinematic viscosity flowing past each other with the same velocity in opposite directions. We also assume that the Alfven velocities and the sound speeds in the two streams are the same. We, therefore have

$$\rho_1 = \rho_2, \quad \gamma_1 = \gamma_2, \quad M_1^2 = M_2^2, \quad C_1^2 = C_2^2, \quad V_1 = V, \quad V_2 = -V. \tag{33}$$

The dispersion relation simplifies considerably for these values of the parameters. Although the model of the streams considered is highly idealized, it is nevertheless hoped that it will reveal the essential features of the effects of viscosity and rotation on the instability of the system. The same model was earlier considered by Singh and Khare [7] for inviscid streams, i. e. when $\gamma_1 = \gamma_2 = 0$.

5. Conclusions

For several values of the parameters characterizing the viscosity and rotation, the dispersion relation has been solved numerically for fixed values of the Alfven velocity and the speed of sound. These calculations are presented in Figs. 1 and 2, where we plot the growth rate (positive root of n) against the wave number k_x for γ (viscosity) = 1.0, 2.0, 3.0, 4.0 and rotation (Ω) = 1.0, 2.0, 3.0 for fixed values of the other parameters.

From Fig. 1 we see that the growth rate decreases with the increase in Ω, showing thereby that rotation has a stabilizing influence on the unstable mode of disturbance. We also see from Fig. 2 that the growth rate decreases with the increase in γ. The viscosity, therefore, suppresses the instability of the system.

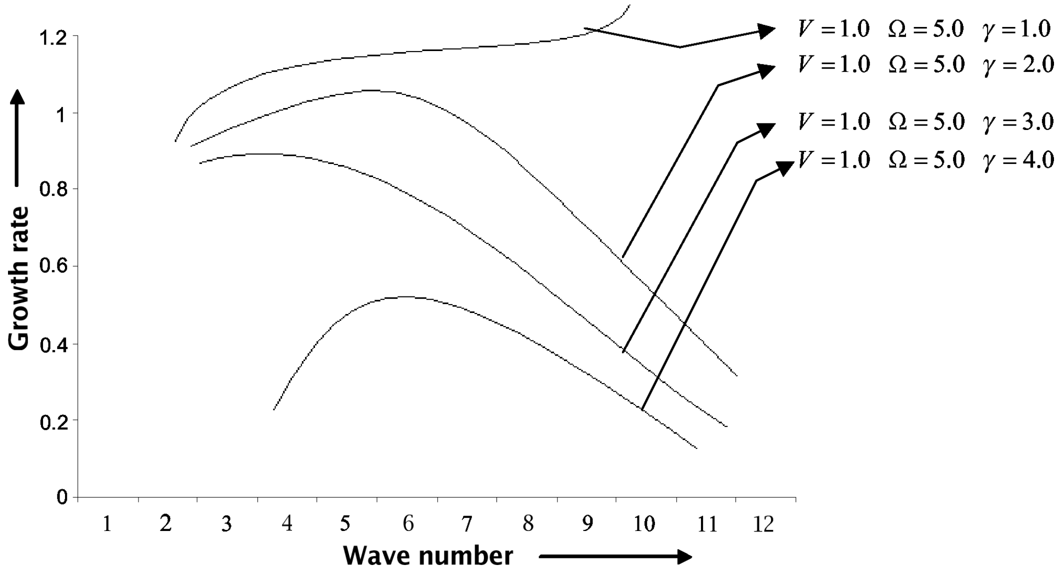

Fig. 2. Plot of the growth rate against the wave number.

We may thus conclude that both the viscosity and rotation suppress the instability of the superposed gravitating streams when the streams rotate about an axis in the horizontal plane which is perpendicular to the direction of the horizontal magnetic field. The results obtained agree with the earlier observations of Agarwal and Bhatia [8].

Acknowledgements

This work was started during the tenure of a UGC Emeritus Fellowship to P.K.B. and was completed under the AICTE Emeritus Fellowship to him. P. K. B. gratefully acknowledges the financial support provided under these fellowships.

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